

Closed strings in uniform magnetic field backgrounds

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Abstract

We consider a class of conformal models describing closed strings in axially symmetric stationary magnetic flux tube backgrounds. These models are closed string analogs of the Landau model of a particle in a magnetic field or the model of an open string in a constant magnetic field. They are interesting examples of solvable unitary conformal string theories with non-trivial 4-dimensional curved space-time interpretation. In particular, their quantum Hamiltonian can be expressed in terms of free fields and the physical spectrum and string partition function can be explicitly determined. In addition to the presence of tachyonic instabilities and existence of critical values of magnetic field the closed string spectrum exhibits also some novel features which were absent in the open string case.

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1. Introduction

The study of behavior of systems of particles and fields in an external uniform magnetic field is one of the basic problems in theoretical physics and has long history. Like temperature, the magnetic field plays the role of a probe which may be used to reveal various properties of a system. A remarkable feature of the uniform magnetic field problems in quantum mechanics and QED is their solvability^{1,2}. That applies to certain extent also to gauge theories in abelian magnetic environment various aspects of which (possibility of restoration of broken symmetry, instability of magnetic background in non-abelian models, formation of vacuum condensates, etc.) were extensively studied in the past³⁻⁶.

It is interesting to try to address a similar problem in the context of string theory (with one of the standard motivations that this may eventually help us to learn about its possible phase structure). The reason why the quantum-mechanical or field-theoretic problem of a particle in a uniform abelian (electro)magnetic field is exactly solvable is that the action $I = \int d\tau [\dot{x}^\mu \dot{x}_\mu + i\dot{x}^\mu A_\mu(x)]$ (which determines the Hamiltonian in quantum mechanics and the heat kernel in field theory) becomes gaussian if the field strength is constant,

$$A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu, \quad F_{\mu\nu} = \text{const}.$$

The same is true also in (abelian) open string theory where the interaction takes place only at the boundary points

$$I = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_a x^\mu \partial^a x_\mu + i \int d\tau A_\mu(x) \dot{x}^\mu,$$

and thus the resulting gaussian path integral can be computed exactly⁷. This is a consistent ‘on-shell’ problem since $F_{\mu\nu} = \text{const}$ is an exact solution of the effective field equations⁷ of the open string theory. Indeed, the corresponding 2d world-sheet theory represents a conformal field theory⁸ which can be solved explicitly in terms of free oscillators thus representing a generalization of the Landau problem in quantum particle mechanics. As a result, one is able to determine the spectrum of an open string moving in a constant magnetic field^{8,9,10}.

A novel feature of this spectrum as compared to the free string spectrum is the presence of new tachyonic states above certain critical values of the magnetic field^{8,10}. That implies that the constant magnetic field background is unstable in the open string theory as it is in the non-abelian gauge theory⁵. The qualitative reason for this instability is that the free open string contains electrically charged higher spin massive particle states. The latter are expected to have (approximately) the Landau spectrum

$$M^2 = M_0^2 + Q\mathcal{H}(2l + 1 - gS),$$

where Q is the charge (the same for all open string states), \mathcal{H} is magnetic field in x_3 direction, g is a gyromagnetic ratio (the effective weak field value of g is 2 for the non-minimally coupled higher spin open string states¹¹), S is the x_3 -component of the spin and $l = 0, 1, 2, \dots$, is the Landau level. Thus M^2 can become negative for large enough values of \mathcal{H} , i.e., $\mathcal{H} > \mathcal{H}_{cr} = M_0^2/Q$ for spin 1 charged states. That applies, for example, to W -bosons in the context of electroweak theory⁶ suggesting the presence of a transition to a phase with a W -condensate (at higher critical field where Higgs field becomes massless the full electroweak symmetry is restored⁶). Note that in the case of unbroken gauge theory with massless charged vector particles the instability is present for any (e.g., infinitesimal) value of the magnetic field⁵. Such ‘infinitesimal’ instability is thus to be expected in the open string theory with *non*-abelian Chan-Paton symmetry (where the constant magnetic field problem does not appear to be easily solvable) and in *closed* string theory (discussed below).

The infra-red instability of a magnetic background is not cured by supersymmetry, i.e. it remains also in supersymmetric gauge theories (e.g., in ultra-violet finite $N = 4$ supersymmetric Yang-Mills theory¹²) since the small fluctuation operator for the gauge field $-\delta_{\mu\nu}D^2 - 2F_{\mu\nu}$ still has negative modes due to the ‘anomalous magnetic moment’ term. This is not surprising given that the magnetic field breaks Lorentz invariance and supersymmetry. This instability is indeed present in the Neveu-Schwarz sector of the open superstring theory¹⁰ (the fermionic Ramond states remain non-tachyonic as in field theory).

Assuming that it is important to try to generalize the open string results to the case of realistic closed models (see, e.g., ref.13) the main question⁷, however, is whether the uniform magnetic field problem is actually tractable in *closed* string theories. An apparent answer is ‘no’ since the abelian vector field must now be coupled to the internal points of the string and such interaction terms, e.g.,

$$L = \partial_a y \partial^a y + A_\mu(x) \partial_a x^\mu \partial^a y + \dots ,$$

in bosonic string or type II superstring (y is a compact internal Kaluza-Klein field that ‘charges’ the string), or

$$L = \bar{\psi} \gamma^a [\partial_a + A_\mu(x) \partial_a x^\mu] \psi + \dots ,$$

in the heterotic string, do not become gaussian for $A_\mu = -\frac{1}{2}F_{\mu\nu}x^\nu$.

One should note, however, that in contrast to the tree level abelian open string case, the $F_{\mu\nu} = \text{const}$ background in flat space does not represent a solution of a *closed* string theory, i.e. the above interaction terms added to the free string Lagrangians do not give conformally invariant 2d σ -models. Indeed, since the closed string theory contains gravity, a uniform magnetic field which has a non-vanishing energy must curve the space (as well as possibly induce other ‘massless’ backgrounds). One should thus first find a consistent

conformal model which is a closed string analog of the uniform magnetic field background in the flat space field (or open string) theory and *then* address the question of its solvability. Remarkably, it turns out^{14–16} that extra terms which should be added to the above closed string actions in order to satisfy the conformal invariance condition (i.e. to satisfy the closed string effective field equations) produce exactly solvable 2d models!

In order to try construct conformal σ -models which can be interpreted as describing closed string in a uniform magnetic field background it is useful to look at possible ‘magnetic’ solutions of low-energy effective string equations. There is a simple analogue of a uniform magnetic field background in the Einstein-Maxwell theory: the static cylindrically symmetric Melvin ‘magnetic universe’ or ‘magnetic flux tube’ solution¹⁷. It has R^4 topology and can be considered¹⁹ as a gravitational analog of the Abrikosov-Nielsen-Olesen vortex¹⁸ with the magnetic pressure (due to repulsion of Faraday’s flux lines) being balanced not by Higgs field but by gravitational attraction. The magnetic field is approximately constant inside the tube and decays to zero at infinity in the direction orthogonal to x_3 -axis. Several interesting features of the Melvin solution in the context of Kaluza-Klein (super)gravity (e.g., instability against monopole or magnetic black hole pair creation) were discussed ref.19 (see also ref.20). This Einstein-Maxwell ($a = 0$) Melvin solution has two straightforward analogs²¹ among solutions of low-energy closed string theory (heterotic string or $D > 4$ bosonic string or superstring toroidally compactified to $D = 4$). In what follows we shall mostly consider the case when the magnetic field has Kaluza-Klein origin. Assuming $x^5 = y$ is a compact internal coordinate, the $D = 5$ string effective action can be expressed in terms of $D = 4$ fields: metric $G_{\mu\nu}$, dilaton ϕ , antisymmetric tensor $B_{\mu\nu}$, two vector fields \mathcal{A}_μ and \mathcal{B}_μ (related to $G_{5\mu}$ and $B_{5\mu}$) and the ‘modulus’ σ . The dilatonic ($a = 1$) and Kaluza-Klein ($a = \sqrt{3}$) Melvin solutions have zero $B_{\mu\nu}$ but ϕ or σ being non-constant.

In addition to the Melvin solutions, the string theory equations admits also another natural uniform magnetic field solution^{22,14} which has $B_{\mu\nu} \neq 0$ and thus has no counterpart in the Einstein-Maxwell theory. It can be considered as a direct closed string analog of the $F_{\mu\nu} = \text{const}$ solution of the Maxwell theory since here the magnetic field is indeed *constant* (and covariantly constant) throughout the space (dilaton is constant as well). Its metric $ds^2 = -(dt + A_i dx^i)^2 + dx_i dx^i + dx_3^2$, $A_i = -\frac{1}{2} F_{ij} x^j$, ($i = 1, 2$) is that of a product of a real line R and the Heisenberg group space H_3 , and the antisymmetric tensor field strength is equal to the constant magnetic field $H_{tij} = F_{ij} = \text{const}$.^a

^a There are also other non-uniform magnetic monopole type string backgrounds which will not be discuss here (see, in particular, ref.23). In addition to the $a = 0$ Melvin solution, another homogeneous magnetic solution of the Einstein-Maxwell theory (which, however, is of less interest since it does not have R^4 topology) is the Robinson-Bertotti one, i.e. $(\text{AdS})_2 \times S^2$ with covariantly constant monopole-type magnetic field $F_{\theta\varphi} = b \sin \theta$ on S^2 . It has an exact string counterpart²⁴ which is a product of the two conformal theories: “ $(\text{AdS})_2$ ” ($\text{SL}(2, \mathbb{R})/\mathbb{Z}$ WZW) and “monopole”²⁵ ($\text{SU}(2)/\mathbb{Z}_m$ WZW) ones. Other monopole-type string solutions were considered in²⁶.

It turns out that the above three basic uniform magnetic field backgrounds (‘constant magnetic field’, ‘ $a = 1$ Melvin’ and ‘ $a = \sqrt{3}$ Melvin’) are exact string solutions to all orders in α' . The conformal $D = 5$ bosonic σ -models which describe them are^{22,14–16} are (the corresponding superstring and heterotic string models^{22,14–16} have similar structure)

$$\begin{aligned} L_{(const)} &= -\partial t \bar{\partial} t + \beta \epsilon_{ij} x^i \bar{\partial} x^j (\partial y - \partial t) + \partial x_i \bar{\partial} x^i + \partial y \bar{\partial} y + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi_0 \\ &= -\partial t \bar{\partial} t + \beta \rho^2 \bar{\partial} \varphi (\partial y - \partial t) + \partial \rho \bar{\partial} \rho + \rho^2 \partial \varphi \bar{\partial} \varphi + \partial y \bar{\partial} y + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi_0 , \end{aligned} \quad (1.1)$$

$$L_{(a=1)} = -\partial t \bar{\partial} t + \partial \rho \bar{\partial} \rho + F(\rho) \rho^2 (\partial \varphi + 2\alpha \partial y) \bar{\partial} \varphi + \partial y \bar{\partial} y + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi(\rho) , \quad (1.2)$$

$$e^{2(\phi - \phi_0)} = F(\rho) = (1 + \alpha^2 \rho^2)^{-1} ,$$

$$L_{(a=\sqrt{3})} = -\partial t \bar{\partial} t + \partial \rho \bar{\partial} \rho + \rho^2 (\partial \varphi + q \partial y) (\bar{\partial} \varphi + q \bar{\partial} y) + \partial y \bar{\partial} y + \partial x_3 \bar{\partial} x_3 + \mathcal{R} \phi_0 . \quad (1.3)$$

Here $x_1 + ix_2 = \rho e^{i\varphi}$, $\varphi \in (0, 2\pi)$ are coordinates of 2-plane orthogonal to the direction of the magnetic field and $y \in (0, 2\pi R)$ is the Kaluza-Klein coordinate used (the charges of string states are proportional to R^{-1}). The the constants α, β, q determine the strength of the abelian magnetic (and other) background fields.

The model (1.1) is a special case of the following model ($u \equiv y - t$, $v \equiv y + t$, $i, j = 1, \dots, D-1$)

$$L = \partial u \bar{\partial} v + \partial x^i \bar{\partial} x^i + 2A_i(x) \bar{\partial} x^i \partial u + \mathcal{R} \phi_0 , \quad (1.4)$$

where the interaction term is reminiscent of the open string coupling. Indeed, (1.4) is conformal to all orders²² if $\partial_i F^{ij} = 0$, i.e., in particular, if $A_i = -\frac{1}{2} F_{ij} x^j$ ((1.1) corresponds to $F_{ij} = \beta \epsilon_{ij}$, $i, j = 1, 2$). The conformal invariance of (1.4) is due to the special chiral ‘null’ structure of the interaction term. When y is non-compact (so that instead of describing a D -dimensional magnetic background (1.4) has $D+1$ -dimensional plane wave interpretation) and $F_{ij} = \text{const}$ (1.4), can be identified with the Lagrangian of the WZW model based on non-semisimple algebra $[e_i, e_j] = F_{ij} e_v$, $[e_i, e_u] = F_{ij} e_j$, $[e_i, e_v] = [e_u, e_v] = 0$ which admits non-degenerate invariant bilinear form, $(e_i, e_j) = \delta_{ij}$, $(e_u, e_v) = \frac{1}{2}$ ((1.1) corresponds to the E_2^c theory of ref.27). The solvability of the constant field model (1.4) or (1.1) is related to the fact that the path integral over v leads to a constraint on u so that the model effectively becomes gaussian in x^i .

Although the models (1.2) and (1.3) look quite different from (1.1), we shall explain below that all of them belong to one 3-parameter (α, β, q) class of string models which are conformally invariant and, moreover, *exactly solvable*¹⁶. They can thus be considered as closed string analogs of the solvable ‘open string in constant magnetic field’ model. In spite of their apparently non-gaussian form they are related (by formal duality transformations) to simpler flat models (this partially is the reason for their solvability). As in the open string case, here one is able to express the corresponding conformal field theory operators in terms of the free creation/annihilation operators and to explicitly determine the

string spectrum^{14,16}. These models appear to be simpler than coset CFT's corresponding to semisimple gauged WZW models (for reviews of solvable (super)string models based on semisimple coset CFT's see, e.g., refs.28,29). For example, their unitarity is easy to demonstrate because of the existence of a light-cone gauge. These models (together with plane-wave type WZW models for non-semisimple groups^{27,30–34}) are thus among the first few known examples of solvable unitary conformal string models with non-trivial $D = 4$ *curved space-time* interpretation.

Below we shall first discuss the target space interpretation of the above models as representing a class of exact stationary axisymmetric magnetic flux tube solutions of string effective equations (Section 2). Then in Section 3 we shall construct the conformal σ -models describing the magnetic flux tube solutions by starting with flat space model and using world-sheet angular duality. This will help to solve the corresponding classical string equations explicitly, expressing the string coordinates in terms of free fields satisfying 'twisted' boundary conditions (Section 4.1). After straightforward operator quantization (Section 4.2) we will find the quantum Virasoro operators. It will then be possible to determine the spectrum of states and partition function (Section 5), in direct analogy with how this is done in simpler models like closed string on a torus or an orbifold, or open string in a constant magnetic field. We shall also discuss some properties of the spectrum, in particular, the two types of tachyonic instabilities present in this closed string model. Some concluding remarks (in particular, about superstring and heterotic string generalizations) will be made in Section 6.

2. Magnetic flux tube solutions of string effective equations

We shall be considering the closed bosonic string (or type II superstring) theory which has no fundamental gauge fields in a higher dimensional space. The abelian gauge fields appear upon toroidal compactification when the theory is 'viewed' from four dimensions. The conformal σ -models which describe $D = 4$ string solutions with non-trivial gauge fields will thus be higher dimensional ones. The simplest case is that of $D = 5$ bosonic string σ -model action (with target space fields not depending on x^5) which can be interpreted as an action of a $D = 4$ string with an internal degree of freedom (compact Kaluza-Klein coordinate x^5) which describes the coupling to additional vector (and scalar) background fields,

$$\begin{aligned}
I_5 &= \frac{1}{\pi\alpha'} \int d^2\sigma [(G_{MN} + B_{MN})(X)\partial X^M \bar{\partial} X^N + \mathcal{R}\phi(X)] \\
&= \frac{1}{\pi\alpha'} \int d^2\sigma [(\hat{G}_{\mu\nu} + B_{\mu\nu})(x)\partial x^\mu \bar{\partial} x^\nu + e^{2\sigma(x)}[\partial y + \mathcal{A}_\mu(x)\partial x^\mu][\bar{\partial} y + \mathcal{A}_\nu(x)\bar{\partial} x^\nu] \\
&\quad + \mathcal{B}_\mu(x)(\partial x^\mu \bar{\partial} y - \bar{\partial} x^\mu \partial y) + \mathcal{R}\phi(x)] ,
\end{aligned} \tag{2.1}$$

where $X^M = (x^\mu, x^5)$, $x^\mu = (t, x^i, x^3)$, $x^5 \equiv y$, $\mathcal{R} \equiv \frac{1}{4}\alpha'\sqrt{\gamma}R^{(2)}$ and

$$\hat{G}_{\mu\nu} \equiv G_{\mu\nu} - G_{55}\mathcal{A}_\mu\mathcal{A}_\nu, \quad G_{55} \equiv e^{2\sigma}, \quad \mathcal{A}_\mu \equiv G^{55}G_{\mu 5}, \quad \mathcal{B}_\mu \equiv B_{\mu 5}. \quad (2.2)$$

From the point of view of the low-energy effective field theory, this decomposition corresponds to starting with the $D = 5$ bosonic string effective action and assuming that one spatial dimension x^5 is compactified on a small circle. Ignoring the massive Kaluza-Klein modes one then finds the following dimensionally reduced $D = 4$ action (see, e.g., ref.35)

$$S_4 = \int d^4x \sqrt{\hat{G}} e^{-2\Phi} \left[\hat{R} + 4(\partial_\mu \Phi)^2 - (\partial_\mu \sigma)^2 \right. \\ \left. - \frac{1}{12}(\hat{H}_{\mu\nu\lambda})^2 - \frac{1}{4}e^{2\sigma}(F_{\mu\nu}(\mathcal{A}))^2 - \frac{1}{4}e^{-2\sigma}(F_{\mu\nu}(\mathcal{B}))^2 + O(\alpha') \right], \quad (2.3)$$

$$F_{\mu\nu}(\mathcal{A}) = 2\partial_{[\mu}\mathcal{A}_{\nu]}, \quad F_{\mu\nu}(\mathcal{B}) = 2\partial_{[\mu}\mathcal{B}_{\nu]}, \quad (2.4)$$

$$\hat{H}_{\lambda\mu\nu} = 3\partial_{[\lambda}B_{\mu\nu]} - 3\mathcal{A}_{[\lambda}F_{\mu\nu]}(\mathcal{B}), \quad \Phi = \phi - \frac{1}{2}\sigma.$$

Given a conformal $D = 5$ σ -model and rewriting its action as in (2.1) one can read off the expressions for the corresponding $D = 4$ background fields which then must represent a solution of the effective equations following from (2.3). These equations have, in particular, the following 3-parameter (α, β, q) class of stationary axisymmetric (electro)magnetic flux tube solutions¹⁶ ($x^\mu = (t, \rho, \varphi, x^3)$)

$$ds_4^2 = -dt^2 + F(\rho)\rho^2(d\varphi - \alpha dt)(d\varphi - \beta dt) \\ - \frac{1}{4}F(\rho)\tilde{F}(\rho)\rho^4[(\alpha - \beta - 2q)d\varphi + q(\alpha + \beta)dt]^2 + d\rho^2 + dx_3^2, \quad (2.5)$$

$$\mathcal{A} = -\frac{1}{2}\tilde{F}(\rho)\rho^2[(\alpha - \beta - 2q)d\varphi + q(\alpha + \beta)dt], \quad (2.6)$$

$$\mathcal{B} = -\frac{1}{2}F(\rho)\rho^2[(\alpha + \beta)d\varphi - (2\alpha\beta + q\alpha - q\beta)dt],$$

$$e^{2(\phi - \phi_0)} = F(\rho), \quad e^{2\sigma} = \frac{F(\rho)}{\tilde{F}(\rho)}, \quad B = -\frac{1}{2}(\alpha - \beta)F(\rho)\rho^2 d\varphi \wedge dt, \quad (2.7)$$

$$F(\rho) \equiv \frac{1}{1 + \alpha\beta\rho^2}, \quad \tilde{F}(\rho) \equiv \frac{1}{1 + q(q + \beta - \alpha)\rho^2}.$$

The metric is stationary and, in general, describes a rotating ‘universe’. For generic values of the parameters the two abelian gauge fields contain both magnetic and electric components with the former being more ‘fundamental’ (there are no solutions when both gauge fields are pure electric). For simplicity we shall call these solutions ‘magnetic flux tube backgrounds’. The three simplest uniform pure magnetic field solutions mentioned in Section 1 are the following special cases (cf.(1.1)–(1.3)): (i) ‘constant magnetic field’ ($q = \alpha = 0$, $\beta \neq 0$):

$$ds_4^2 = -(dt + \frac{1}{2}\beta\rho^2 d\varphi)^2 + d\rho^2 + \rho^2 d\varphi^2 + dx_3^2, \quad (2.8)$$

$$\mathcal{A} = -\mathcal{B} = \frac{1}{2}\beta\rho^2 d\varphi, \quad \sigma = \phi - \phi_0 = 0, \quad B = \frac{1}{2}\beta\rho^2 d\varphi \wedge dt, \quad \hat{H}_{tij} = F_{ij},$$

(ii) ‘ $a = 1$ Melvin’ ($\alpha = \beta = q \neq 0$):

$$ds_4^2 = -dt^2 + d\rho^2 + F^2(\rho)\rho^2 d\varphi^2 + dx_3^2, \quad (2.9)$$

$$\mathcal{A} = -\mathcal{B} = \alpha F(\rho)\rho^2 d\varphi, \quad B = 0, \quad \sigma = 0, \quad e^{2(\phi-\phi_0)} = F = (1 + \alpha^2 \rho^2)^{-1},$$

(iii) ‘ $a = \sqrt{3}$ Melvin’ ($\alpha = \beta = 0, q \neq 0$):

$$ds_4^2 = -dt^2 + d\rho^2 + \tilde{F}(\rho)\rho^2 d\varphi^2 + dx_3^2, \quad (2.10)$$

$$\mathcal{A} = q\tilde{F}(\rho)\rho^2 d\varphi, \quad \mathcal{B} = 0, \quad B = 0, \quad \phi = \phi_0, \quad e^{2\sigma} = \tilde{F}^{-1} = 1 + q^2 \rho^2.$$

In addition to the $q = 0$ subclass of pure magnetic backgrounds (where \mathcal{A} has constant field strength) which generalize (2.8), there are two other special subclasses: $\alpha = q$ (stationary metric, non-zero $B_{\mu\nu}$, zero σ) and $\alpha = \beta$ (static metric, zero $B_{\mu\nu}$, non-zero σ). Solutions with $\alpha\beta \geq 0$, $q(q + \beta - \alpha) \geq 0$ have no curvature singularities.

The above leading-order solutions (2.5)–(2.7) are actually exact to all orders in α' since it turns out that they correspond (according to (2.1)) to conformal $D = 5$ σ -models discussed in the next section.^b

3. Conformal string models describing flux tube backgrounds

The three conformal $D = 5$ σ -models that correspond to the $D = 4$ solutions (2.8),(2.9),(2.10) according to (2.1) are indeed (1.1),(1.2),(1.3). All three models have free-theory central charge. In the case of non-compact y , i.e. in the limit $R \rightarrow \infty$, they are equivalent to other known models. The constant field model (1.1) becomes the ‘plane-wave’ E_2^c WZW model²⁷ with the corresponding CFT discussed in refs.31,14,36. The $a = 1$ Melvin model¹⁵ (1.2) with coordinates formally taken to be non-compact can be identified with a particular limit of $[SL(2, R) \times R]/R$ gauged WZW (‘black string’³⁷) model^c or, equivalently, with the $E_2^c/U(1)$ coset theory³². The $R = \infty$ case of the $a = \sqrt{3}$ Melvin model (1.3) is identical to the flat space model after the redefinition of φ .

The solvability of the general 3-parameter (α, β, q) class of string models corresponding to (2.5)–(2.7) can be understood by using their relation via duality and formal coordinate

^b This class of solutions was actually found¹⁶ not by solving the complicated equations which follow from (2.3) but by explicitly constructing the corresponding $D = 5$ conformal σ -model discussed below. Similar approach to constructing exact string solutions was used in ref.22.

^c In this limit $k \rightarrow \infty$ and the mass and charge of ‘black string’ vanish but simultaneous rescalings of coordinates give rise to a nontrivial model.

shifts to flat space models. Consider, for example, the σ -model which a direct product of $D = 2$ Minkowski space and $D = 2$ ‘dual 2-plane’

$$\tilde{I} = \frac{1}{\pi\alpha'} \int d^2\sigma [\partial u \bar{\partial} v + \partial \rho \bar{\partial} \rho + \rho^{-2} \partial \tilde{\varphi} \bar{\partial} \tilde{\varphi} + \mathcal{R}(\phi_0 - \ln \rho)] . \quad (3.1)$$

$\tilde{\varphi}$ should have period $2\pi\alpha'$ to preserve equivalence of the ‘dual 2-plane’ model to the flat 2-plane CFT³⁸, i.e. to the flat space model^d

$$I_0 = \frac{1}{\pi\alpha'} \int d^2\sigma (\partial u \bar{\partial} v + \partial \rho \bar{\partial} \rho + \rho^2 \partial \hat{\varphi} \bar{\partial} \hat{\varphi} + \mathcal{R}\phi_0) . \quad (3.2)$$

If we now make coordinate shifts and add a constant antisymmetric tensor term we obtain from (3.1) (α, β, q are free parameters of dimension cm^{-1})

$$\begin{aligned} \tilde{I} = \frac{1}{\pi\alpha'} \int d^2\sigma [(\partial u + \alpha \partial \tilde{\varphi})(\bar{\partial} v + \beta \bar{\partial} \tilde{\varphi}) + \partial \rho \bar{\partial} \rho + \rho^{-2} \partial \tilde{\varphi} \bar{\partial} \tilde{\varphi} \\ + \frac{1}{2} q [\partial(u + v) \bar{\partial} \tilde{\varphi} - \bar{\partial}(u + v) \partial \tilde{\varphi}] + \mathcal{R}(\phi_0 - \ln \rho)] . \end{aligned} \quad (3.3)$$

The two models (3.1) and (3.3) are of course ‘locally-equivalent’; in particular, (3.3) also solves the conformal invariance equations. However, if u and v are periodic, i.e. if u and v are given by

$$u \equiv y - t , \quad v \equiv y + t , \quad y \in (0, 2\pi R) , \quad (3.4)$$

then the ‘shifted’ coordinates $u + \alpha \tilde{\varphi}$ and $v + \beta \tilde{\varphi}$ are not globally defined for generic α and β (the periods of $y = \frac{1}{2}(u + v)$ and φ are different) and the torsion term is non-trivial for $q \neq 0$. As a result, the conformal field theories corresponding to (3.1) and (3.3) will *not* be equivalent. The $O(3, 3; R)$ duality transformation with *continuous* coefficients which relates the model (3.3) to the flat space one (3.2) is not a symmetry of the flat CFT, i.e. leads to a new conformal model which, however, is simple enough to be explicitly solvable¹⁶.

Starting with (3.3) and making the duality transformation in $\tilde{\varphi}$ one obtains a more complicated σ -model ($\pi\alpha' I \equiv \int d^2\sigma L$)

$$\begin{aligned} L = F(\rho)(\partial u - \alpha \rho^2 \partial \varphi')(\bar{\partial} v + \beta \rho^2 \bar{\partial} \varphi') + \partial \rho \bar{\partial} \rho + \rho^2 \partial \varphi' \bar{\partial} \varphi' \\ + \partial x_3 \bar{\partial} x_3 + \mathcal{R}(\phi_0 + \frac{1}{2} \ln F) , \end{aligned} \quad (3.5)$$

^d The two models are equivalent in the sense of a relation of classical solutions and equality of the correlators of certain operators (e.g., $\partial \tilde{\varphi}$ and $\rho^2 \partial \varphi$) but the spectra of states are formally different (see also ref.39): the spectrum is continuous on 2-plane and discrete on dual 2-plane (with duality relating states with given orbital momentum on 2-plane and states with given winding number on dual 2-plane).

$$F^{-1} = 1 + \alpha\beta\rho^2 , \quad \varphi' \equiv \varphi + \frac{1}{2}q(u+v) .$$

Here $\varphi \in (0, 2\pi)$ is the periodic coordinate dual to $\tilde{\varphi}$ and we have also added a free x_3 -coordinate term. Since the periods of φ and $y = \frac{1}{2}(u+v)$ are, in general, different φ' is not globally defined. The theory (3.5) is conformally invariant to all orders in α' . For the purpose of demonstrating this one may ignore the difference between φ' and φ (i.e. may set $q = 0$ or consider y to be non-compact). Then (3.5) becomes equivalent to a special case of the ‘generalized F -model’ which was shown to be conformally invariant²².

It is the σ -model (3.5) that defines the string theory corresponding to the class of $D = 4$ magnetic flux tube backgrounds (2.5)–(2.7). The models (1.1), (1.2), (1.3) are the special cases of (3.5): $\alpha = q = 0$, $\alpha = \beta = q$ and $\alpha = \beta = 0$.

4. Solution of the string models

The relation of the class of string models (3.5) to the flat space model via formal duality and coordinate shifts makes possible to solve the classical string equations (which look quite complicated) explicitly. The fact that the two dual models have related classical solutions enables to express the solution in terms of free fields satisfying ‘twisted’ boundary conditions^{14,16}. One can then proceed to straightforward operator quantization (fixing, e.g., a ‘light-cone’ gauge). Some of the resulting expressions are similar to those appearing in the simpler cases of open string theory in a constant magnetic field⁸ or R^2/Z_N orbifold model⁴⁰.

4.1. Solution of the classical equations on the cylinder

Introducing the free field $X = X_1 + iX_2$ such that

$$\begin{aligned} L_0 &= \partial_+\rho\partial_-\rho + \rho^2\partial_+\hat{\varphi}\partial_-\hat{\varphi} = \partial_+X\partial_-X^* , \quad X \equiv \rho e^{i\hat{\varphi}} , \\ \rho^2 &= XX^* , \quad \hat{\varphi} = \frac{1}{2i} \ln \frac{X}{X^*} , \quad X = X_+(\sigma_+) + X_-(\sigma_-) , \quad \sigma_{\pm} = \tau \pm \sigma , \end{aligned} \quad (4.1)$$

we can represent the solution of equations following from (3.1) in the form

$$\begin{aligned} \partial_{\pm}\tilde{\varphi} &= \mp\rho^2\partial_{\pm}\hat{\varphi} = \pm\frac{i}{2}(X^*\partial_{\pm}X - X\partial_{\pm}X^*) , \\ \tilde{\varphi}(\sigma, \tau) &= 2\pi\alpha'[J_-(\sigma_-) - J_+(\sigma_+)] + \frac{i}{2}(X_+X_-^* - X_+^*X_-) , \\ J_{\pm}(\sigma_{\pm}) &\equiv \frac{i}{4\pi\alpha'} \int_0^{\sigma_{\pm}} d\sigma_{\pm} (X_{\pm}\partial_{\pm}X_{\pm}^* - X_{\pm}^*\partial_{\pm}X_{\pm}) . \end{aligned} \quad (4.2)$$

Then the solution of the string equations corresponding to (3.5) is

$$u = U_+ + U_- - \alpha\tilde{\varphi} , \quad v = V_+ + V_- - \beta\tilde{\varphi} , \quad x \equiv \rho e^{i\varphi} = e^{-iq(u+v)} e^{i\alpha V_- - i\beta U_+} X , \quad (4.3)$$

where U_{\pm} and V_{\pm} are arbitrary functions of σ_{\pm} . The closed string boundary condition on the cylinder $x(\sigma + \pi, \tau) = x(\sigma, \tau)$ implies that the free field $X = X_+ + X_-$ must satisfy the “twisted” condition

$$X(\sigma + \pi, \tau) = e^{i\gamma\pi} X(\sigma, \tau) , \quad X_{\pm} = e^{\pm i\gamma\sigma_{\pm}} \mathcal{X}_{\pm} , \quad \mathcal{X}_{\pm}(\sigma_{\pm} \pm \pi) = \mathcal{X}_{\pm}(\sigma_{\pm}) , \quad (4.4)$$

where $\mathcal{X}_{\pm} = \mathcal{X}_{\pm}(\sigma_{\pm})$ are free fields with standard periodic boundary conditions

$$\mathcal{X}_+ = i\sqrt{\alpha'/2} \sum_{n=-\infty}^{\infty} \tilde{a}_n \exp(-2in\sigma_+) , \quad \mathcal{X}_- = i\sqrt{\alpha'/2} \sum_{n=-\infty}^{\infty} a_n \exp(-2in\sigma_-) . \quad (4.5)$$

Since $y = \frac{1}{2}(u + v)$ is compactified on a circle of radius R ,

$$u(\sigma + \pi, \tau) = u(\sigma, \tau) + 2\pi w R , \quad v(\sigma + \pi, \tau) = v(\sigma, \tau) + 2\pi w R , \quad w = 0, \pm 1, \dots,$$

where w is the winding number. As a result,

$$U_{\pm} = \sigma_{\pm} p_{\pm}^u + U'_{\pm} , \quad V_{\pm} = \sigma_{\pm} p_{\pm}^v + V'_{\pm} , \quad (4.6)$$

$$p_{\pm}^u = \pm(wR - \alpha'\alpha J) + p_u , \quad p_{\pm}^v = \pm(wR - \alpha'\beta J) + p_v ,$$

where U'_{\pm} and V'_{\pm} are single-valued functions of σ_{\pm} , p_u and p_v are arbitrary parameters (related to the Kaluza-Klein momentum and the energy of the string) and J is the angular momentum ($J_{L,R} \equiv J_{\pm}(\pi)$)

$$J = J_R + J_L = -\frac{1}{2} \sum_n (n + \frac{1}{2}\gamma) a_n^* a_n - \frac{1}{2} \sum_n (n - \frac{1}{2}\gamma) \tilde{a}_n^* \tilde{a}_n . \quad (4.7)$$

Then the ‘twist’ parameter γ in (4.4) is given by

$$\gamma = (2q + \beta - \alpha)wR + \beta p_u + \alpha p_v . \quad (4.8)$$

Evaluating the classical stress-energy tensor on the above solution one finds that it takes the “free-theory” form $T_{\pm\pm} = \partial_{\pm} U_{\pm} \partial_{\pm} V_{\pm} + \partial_{\pm} X \partial_{\pm} X^*$. It is convenient to fix the light-cone gauge, using the remaining conformal symmetry to gauge away the ‘non zero-mode’ parts U'_{\pm} of U . Then the classical constraints $T_{--} = T_{++} = 0$ can be solved as usual and determine V'_{\pm} in terms of the free fields X_{\pm} . The classical expressions for Virasoro operators $L_0 = \frac{1}{4\pi\alpha'} \int d\sigma T_{--}$, $\tilde{L}_0 = \frac{1}{4\pi\alpha'} \int d\sigma T_{++}$ are

$$L_0 = \frac{p_-^u p_-^v}{4\alpha'} + \frac{1}{2} \sum_n \left(n + \frac{1}{2}\gamma\right)^2 a_n^* a_n , \quad \tilde{L}_0 = \frac{p_+^u p_+^v}{4\alpha'} + \frac{1}{2} \sum_n \left(n - \frac{1}{2}\gamma\right)^2 \tilde{a}_n^* \tilde{a}_n . \quad (4.9)$$

4.2. Quantum Virasoro operators

One can now quantize the theory by imposing the standard commutation relations between canonical momenta and coordinates $[P_x(\sigma, \tau), x^*(\sigma', \tau)] = -i\delta(\sigma - \sigma')$, etc. Because of the duality between (3.3) and (3.5) this turns out to be the same as demanding the canonical commutation relations for the fields X, X^* of the free (but globally non-trivial, cf.(4.4)) theory. As a result, p_u, p_v and the Fourier modes a_n, \tilde{a}_n become operators in a Hilbert space. One finds that

$$[a_n, a_m^*] = 2(n + \frac{1}{2}\gamma)^{-1}\delta_{nm} , \quad [\tilde{a}_n, \tilde{a}_m^*] = 2(n - \frac{1}{2}\gamma)^{-1}\delta_{nm} . \quad (4.10)$$

p_u, p_v and thus γ (4.8) commute with a_n, \tilde{a}_n and can be expressed in terms of the conserved string energy $E = \int_0^\pi d\sigma P_t$ and the quantized Kaluza-Klein linear momentum $p_y = \int_0^\pi d\sigma P_y = m/R$, $m = 0, \pm 1, \dots$,

$$E = \frac{1}{2\alpha'}[p_u - p_v - \alpha'(\alpha + \beta)\hat{J}] , \quad p_y = \frac{1}{2\alpha'}[p_u + p_v + \alpha'(2q + \beta - \alpha)\hat{J}] , \quad (4.11)$$

$$\gamma = (2q + \beta - \alpha)wR + \alpha'[(\alpha + \beta)mR^{-1} - (\alpha - \beta)E] - \frac{1}{2}\alpha'q(\alpha + \beta)\hat{J} . \quad (4.12)$$

\hat{J} is the angular momentum operator obtained by ‘symmetrizing’ the classical J (4.7). The quantum Virasoro operators \hat{L}_0 and $\hat{\tilde{L}}_0$ (and thus the quantum Hamiltonian $\hat{H} = \hat{L}_0 + \hat{\tilde{L}}_0$) are then given by symmetrized expressions in (4.9).^e

The sectors of states can be labeled by the conserved quantum numbers: the energy E , the angular momentum \hat{J} in the x_1, x_2 plane, and the linear m/R and winding wR Kaluza-Klein momenta or “charges” (and also by momenta in additional 22 spatial dimensions which we shall add to saturate the central charge condition). As in the case of the Landau model or the open string model⁸, the states with generic values of γ are “trapped” by the magnetic field. The states in the “hyperplanes” in the (m, w, E, \hat{J}) space with $|\gamma| = 2n$, $n = 0, 1, \dots$, are special: for them the translational invariance on the (x_1, x_2) -plane is restored with the zero-mode oscillators $a_0, a_0^*, \tilde{a}_0, \tilde{a}_0^*$ being replaced by the zero mode part of the coordinate x and conjugate linear momentum.

Restricting the consideration to the sector of states with $0 < \gamma < 2$ one can introduce the normalized creation and annihilation operators ($n, m = 1, 2, \dots$)

$$[b_{n\pm}, b_{m\pm}^\dagger] = \delta_{nm}, \quad [\tilde{b}_{n\pm}, \tilde{b}_{m\pm}^\dagger] = \delta_{nm}, \quad [b_0, b_0^\dagger] = 1, \quad [\tilde{b}_0, \tilde{b}_0^\dagger] = 1, \quad (4.13)$$

^e In agreement with the defining relations in (4.4) the expressions for \hat{H}, \hat{J} and the commutation relations (4.10) are invariant under $\gamma \rightarrow \gamma + 2$ combined with the corresponding renaming of the mode operators $a_n \rightarrow a_{n+1}, \tilde{a}_n \rightarrow \tilde{a}_{n-1}$.

$$b_{n+}^\dagger = a_{-n}\omega_-, \quad b_{n-} = a_n\omega_+, \quad b_0 = \frac{1}{2}\sqrt{\gamma}a_0, \quad \tilde{b}_0^\dagger = \frac{1}{2}\sqrt{\gamma}\tilde{a}_0, \quad \omega_\pm \equiv \sqrt{\frac{1}{2}(n \pm \frac{1}{2}\gamma)}.$$

The Hamiltonian and Virasoro operators then take the form¹⁶

$$\hat{H} = \hat{L}_0 + \hat{\tilde{L}}_0 = \frac{1}{2}\alpha'(-E^2 + p_a^2 + \frac{1}{2}Q_+^2 + \frac{1}{2}Q_-^2) + N + \tilde{N} - 2c_0 \quad (4.14)$$

$$\begin{aligned} & -\alpha'[(q+\beta)Q_+ + \beta E]J_R - \alpha'[(q-\alpha)Q_- + \alpha E]J_L \\ & + \frac{1}{2}\alpha'q[(q+2\beta)J_R^2 + (q-2\alpha)J_L^2 + 2(q+\beta-\alpha)J_RJ_L] , \\ & \hat{L}_0 - \hat{\tilde{L}}_0 = N - \tilde{N} - mw . \end{aligned} \quad (4.15)$$

Here Q_\pm are the left and right combinations of the Kaluza-Klein linear and winding momenta (which play the role of charges in the present context), c_0 is the normal ordering term^f

$$Q_\pm \equiv \frac{m}{R} \pm \frac{wR}{\alpha'}, \quad c_0 \equiv 1 - \frac{1}{4}\gamma(1 - \frac{1}{2}\gamma),$$

and p_a , $a = 3, \dots, 24$ are momenta in additional free spatial dimensions. The operators N, \tilde{N} and the angular momentum operators J_L, J_R have the standard ‘free-theory’ form (a_{na}, \tilde{a}_{na} correspond to free spatial directions, $a = 3, \dots, 24$)

$$\begin{aligned} N &= \sum_{n=1}^{\infty} n(b_{n+}^\dagger b_{n+} + b_{n-}^\dagger b_{n-} + a_{na}^\dagger a_{na}), \quad \tilde{N} = \sum_{n=1}^{\infty} n(\tilde{b}_{n+}^\dagger \tilde{b}_{n+} + \tilde{b}_{n-}^\dagger \tilde{b}_{n-} + \tilde{a}_{na}^\dagger \tilde{a}_{na}), \\ \hat{J}_R &= -b_0^\dagger b_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (b_{n+}^\dagger b_{n+} - b_{n-}^\dagger b_{n-}) \equiv J_R - \frac{1}{2} \rightarrow -l_R - \frac{1}{2} + S_R, \\ \hat{J}_L &= \tilde{b}_0^\dagger \tilde{b}_0 + \frac{1}{2} + \sum_{n=1}^{\infty} (\tilde{b}_{n+}^\dagger \tilde{b}_{n+} - \tilde{b}_{n-}^\dagger \tilde{b}_{n-}) \equiv J_L + \frac{1}{2} \rightarrow l_L + \frac{1}{2} + S_L, \quad \hat{J} = J_R + J_L = J. \end{aligned} \quad (4.16)$$

The analogs of the above expressions in the sectors with $2k < \gamma < 2k + 2$, $k = \text{integer}$, can be found in a similar way by ‘renaming’ the creation and annihilation operators. The result is the same as in (4.14) with the replacement $\gamma \rightarrow \gamma' = \gamma - 2k$ in c_0 .

It is remarkable that the complicated space-time background (2.5)–(2.7) is associated with relatively simple CFT described by (4.14). The first line in (4.14) with $c_0 \rightarrow 1$ is the Hamiltonian of a free closed string compactified on a circle. The second line (together with $O(J)$ term in c_0 , see (4.12)) is the analogue of the gyromagnetic interaction term for

^f The normal ordering constant is fixed by the Virasoro algebra. The free-string constant in \hat{L}_0 is shifted from 1 to c_0 . This corresponds to computing infinite sums using the generalized ζ -function regularization. Similar shift is found in the open string theory⁸ and in orbifold models and is characteristic to the case of a free boson with twisted boundary conditions. It is also consistent with modular invariance of the partition function.

a particle in a magnetic field.^g Similar term is present in the Hamiltonian of the open string in a constant magnetic field⁸

$$\hat{H}^{(open)} = L_0 = \frac{1}{2}\alpha'(-E^2 + p_a^2) + N - c_0(\gamma) - \gamma J_R, \quad (4.17)$$

$$c_0 = 1 - \frac{1}{4}\gamma(1 - \frac{1}{2}\gamma), \quad \gamma \equiv \frac{2}{\pi}|\arctan(2\alpha'\pi Q_1\beta) + \arctan(2\alpha'\pi Q_2\beta)|,$$

where Q_1, Q_2 are charges at the two ends of the open string, N and \hat{J}_R have the same form as in (4.16) and β is proportional to the magnetic field, $F_{ij} = \beta\epsilon_{ij}$. The $O(J^2)$ terms in the third line of (4.14) (and in c_0) are special to closed string theory.^h

In the special cases corresponding to the ‘constant magnetic field’ model (1.1), (2.8), and the $a = 1$ and $a = \sqrt{3}$ Melvin models (1.2), (2.9) and (1.3), (2.10) the Hamiltonian (4.14) takes the following form

$$\alpha = q = 0 : \quad \hat{H} = \hat{H}_0 - 2c_0(\gamma) - \alpha'\beta(Q_+ + E)J_R, \quad \gamma = \alpha'\beta(Q_+ + E), \quad (4.18)$$

$$\hat{H}_0 \equiv \frac{1}{2}\alpha'(-E^2 + p_a^2 + \frac{1}{2}Q_+^2 + \frac{1}{2}Q_-^2) + N + \tilde{N},$$

$$\alpha = \beta = q : \quad \hat{H} = \hat{H}_0 - 2c_0(\gamma) - 2\alpha'\alpha Q_+ J_R - \alpha'\alpha E(J_R + J_L) \quad (4.19)$$

$$+ \frac{1}{2}\alpha'\alpha^2(J_R + J_L)(3J_R - J_L), \quad \gamma = \alpha'\alpha[Q_+ - \alpha(J_R + J_L)],$$

$$\alpha = \beta = 0 : \quad \hat{H} = \hat{H}_0 - 2c_0(\gamma) - \alpha'q(Q_+ J_R + Q_- J_L) + \frac{1}{2}\alpha'q^2(J_R + J_L)^2, \quad (4.20)$$

$$\gamma = 2qwR.$$

Note that presence of the $O(\gamma^2)$ normal ordering term in c_0 in (4.14) implies (see (4.12)) that the quantum Hamiltonians (4.14), (4.18)–(4.20) contain just one $O(\alpha'^2)$ term which is of higher order in α' than other ‘semiclassical’ terms.ⁱ

5. String spectrum and partition function

Using (4.14), (4.15) to define the Virasoro constraints

$$\hat{L}_0 = \hat{\tilde{L}}_0 = 0 \quad \rightarrow \quad \hat{H} = 0, \quad N - \tilde{N} = mw, \quad (5.1)$$

^g The $EJ_{L,R}$ and $\alpha'E^2$ terms (explicit in (4.14) and implicit in c_0 through its dependence on γ) reflect the non-static nature of corresponding subclass of backgrounds (which in turn is related to the presence of the non-vanishing antisymmetric tensor).

^h It is clear from the above construction that (4.14) is, at the same time, also the Hamiltonian for the φ -dual theory (3.3). The origin of the J^2 terms in \hat{H} can be traced, in particular, to the presence of the $\alpha\beta\partial\tilde{\varphi}\bar{\partial}\tilde{\varphi}$ term in (3.3).

ⁱ The presence of this higher order term is consistent with current algebra approaches in the two special cases when our model becomes equivalent to a WZW or coset model: (i) $R = \infty$ limit of the constant magnetic field model (1.1) which is equivalent to the E_2^c WZW model^{27,31,32} (for which the quantum stress tensor contains order $1/k$ correction equivalent to the term in (4.18) in this limit); (ii) the non-compact limit of the $a = 1$ Melvin model (1.2) which is related to a special limit of the $SL(2, R) \times R/R$ gauged WZW model, or to the $E_2^c/U(1)$ coset theory, the Hamiltonian of which also contains $1/k$ correction term³².

it is straightforward to compute the spectrum of states^{14,16} just as this is done in the free string theory. Indeed, even though the Hamiltonian (4.14) containing $O(J^2)$ terms is, in general, of *fourth* order in creation and annihilation operators, it is *diagonal* in Fock space since N, \tilde{N}, J_L and J_R have diagonal form. The continuous momenta $p_{1,2}$ corresponding to the zero modes of the coordinates $x_{1,2}$ of the plane are effectively replaced by the integer eigenvalues $l_R, l_L = 0, 1, 2, \dots$ of the zero-mode parts $b_0^\dagger b_0$ and $\tilde{b}_0^\dagger \tilde{b}_0$ of \hat{J}_R and \hat{J}_L (see (4.16)). Thus the ‘2-plane’ part of the spectrum is discrete in the $0 < \gamma < 2$ sector (but, as mentioned above, becomes continuous when $\gamma = 0$ or $\gamma = 2$).^j Generic string states are thus ‘trapped’ by the magnetic flux tube as in the Landau problem or the open string case. This is consistent with a picture of a charged closed string moving in magnetic field orthogonal to the plane.

For example, let us consider the scalar state at zero string excitation level $S_L = S_R = N = \tilde{N} = 0$ in the non-winding ($w = 0$) sector. The eigen-values of \hat{J}_R and \hat{J}_L in (4.16) are $-l_R - \frac{1}{2}$ and $l_L + \frac{1}{2}$ ($l_{L,R} = 0, 1, 2, \dots$ are the analogs of the Landau level). Then in the $a = 1$ Melvin model (4.19) $\hat{H} = 0$ reduces to

$$M^2 \equiv E^2 - p_a^2 = -4\alpha'^{-1} + p_y^2 + 2\alpha p_y(2l_R + 1) - 2\alpha^2(l_L - l_R)(2l_R + 1) \quad (5.2)$$

$$-2\alpha'\alpha^2[p_y - \beta(l_L - l_R)]^2, \quad p_y = m/R,$$

where it is assumed that $0 < \gamma = 2\alpha'\alpha[p_y - \alpha(l_L - l_R)] < 2$. The same spectrum (up to the $O(\alpha')$ term coming from γ^2 in c_0 in (4.14)) can be found by directly solving the tachyon equation (to the leading order in α')

$$\alpha'[\Delta + O(\alpha')]T = 4T, \quad \Delta = -\frac{1}{\sqrt{-G}e^{-2\phi}}\partial_\mu(\sqrt{-G}e^{-2\phi}G^{\mu\nu}\partial_\nu).$$

In the $D = 5$ background corresponding to the Melvin model (1.2),

$$[-\partial_t^2 + \rho^{-1}\partial_\rho(\rho\partial_\rho) + \rho^{-2}(1 + \alpha^2\rho^2)^2\partial_\varphi^2 + (1 + \alpha^2\rho^2)\partial_y(\partial_y - 2\alpha\partial_\varphi)]T = -4\alpha'^{-1}T, \quad (5.3)$$

and (5.2) is reproduced by taking $T = \exp(iEt + ip_y y + il\varphi) \tilde{T}(\rho)$, $l = l_L - l_R$. Similar correspondence between the string spectrum and the solution of the tachyon equation is found also in the constant magnetic field model (1.1) where the point-particle limit of the Hamiltonian (4.18) is

$$\hat{H}_0 = \frac{1}{2}\alpha'[-E^2 + p_a^2 + p_y^2 + 2(p_y + E)\beta(l_R + \frac{1}{2}) - \frac{1}{2}\alpha'\beta^2(p_y + E)^2] - 2. \quad (5.4)$$

^j The Hamiltonian for the case of $\gamma = 0$ is obtained by adding $\frac{1}{2}\alpha'(p_1^2 + p_2^2)$ and replacing $-b_0^\dagger b_0 - \frac{1}{2}$ and $\tilde{b}_0^\dagger \tilde{b}_0 + \frac{1}{2}$ in \hat{J}_R and \hat{J}_L in (4.16) by one half of the center of mass orbital momentum $(x_1 p_2 - x_2 p_1)$.

In semiclassical approximation (5.4) is similar to the Landau Hamiltonian with $p_y + E$ playing the role of charge. The unusual dependence on the energy is due to the fact that the background (2.8) is stationary but not static (or, equivalently, due to the presence of the antisymmetric tensor background as demanded by conformal invariance). For the subclass of models with $\alpha = \beta$ the metric is static (and $B_{\mu\nu} = 0$, see (2.5),(2.7)) and we get more direct correspondence with familiar particle theory expressions. Indeed, in the weak-field limit when $\alpha = \beta$ and q are small one finds from (4.14) (cf. (5.2))

$$M^2 = M_0^2 - 2(q_+ Q_+ S_R + q_- Q_- S_L) + [(2l_R + 1)q_+ Q_+ - (2l_L + 1)q_- Q_-] + O(q_\pm^2) ,$$

$$\alpha' M_0^2 = -4 + 2N + 2\tilde{N} + \frac{1}{2}Q_+^2 + \frac{1}{2}Q_-^2 , \quad q_\pm = q \pm \alpha .$$

Since in the *closed* string models we consider there are *two* $U(1)$ gauge fields (2.6) with strengths determined by q_\pm we get two gyromagnetic ratios (in general, different from 2), $g_R = 2S_R/S$, $g_L = 2S_L/S$, $S = S_R + S_L$, which are in agreement¹⁶ with earlier suggestions ref.13.

5.1. Tachyonic instabilities

The effect that the magnetic field produces on the energy of a generic state can be interpreted as a combination of the gyromagnetic Landau-type interaction and the influence of curved space-time geometry. The important property of the spectrum is the appearance of new tachyonic instabilities, typically associated with states with angular momentum aligned along the magnetic field. Similar magnetic instabilities were found in non-abelian field theories^{5,6} (where one has charged spin 1 particles with non-minimal coupling) and in the (abelian) open string theory⁸. In the open string case there is a sequence of critical values of the magnetic field for which highest spin component states (lying on the first Regge trajectory, $(b_{1+}^\dagger)^k |0; l_R = 0 \rangle$, cf. (4.16),(4.17)) become tachyonic¹⁰. The new feature of the closed string theory is the existence of states with arbitrarily large charges. Since the critical magnetic field at which a given state of a charge Q may become tachyonic is of order of $1/(\alpha' Q)$ there is an infinite number of tachyonic instabilities for any given finite value of the magnetic field. Also, in contrast to the abelian open (super)string model^{8,10} where charged $S \geq 1$ spin states are massive and thus instability can appear only for large ($\sim 1/\alpha'$) values of the magnetic field, the free closed string spectrum contains charged massless states so that (as in the unbroken gauge theory⁵) tachyonic states appear for an infinitesimal value of the background magnetic field.

To illustrate the presence of the new tachyonic instabilities let us consider the constant magnetic field model (4.18) and look at the states which complete the $SU(2)_R$ massless vector multiplet in the free ($\beta = 0$) theory compactified on a circle of ‘self-dual’ radius

$R = \sqrt{\alpha'}$. The states with $S_R \neq 0$ are $b_{1\pm}^\dagger |0; m = w = 1\rangle$, $b_{1\pm}^\dagger |0; m = w = -1\rangle$, i.e. have $\tilde{N} = 0$, $J_R = -l_R \pm 1$, and the energy

$$\kappa [E + \kappa^{-1} \beta (\hat{J}_R + \frac{1}{2} \alpha' \beta Q_+)]^2 = -4\alpha'^{-1} + \kappa^{-1} (Q_+ - \beta \hat{J}_R)^2, \quad (5.5)$$

$$\kappa \equiv 1 + \frac{1}{2} \alpha' \beta^2, \quad Q_+ = (r + r^{-1}) / \sqrt{\alpha'}, \quad r \equiv R / \sqrt{\alpha'}.$$

If $r = 1$, an infinitesimal magnetic field $\beta > 0$ makes the component with $J_R = 1$ tachyonic. This instability is the same as in the non-abelian gauge theory⁵. Away from the self-dual radius, this state is mahas real energy for small β and becomes tachyonic for some finite critical value β_{cr} of the magnetic field.

Instabilities caused by the linear in $\hat{J}_{L,R}$ terms in \hat{H} are present also in the $\alpha = \beta$ models, in particular, in the $a = 1$ Melvin model (4.19). For example, the mass of the level one state with $w = 0$, $m > 0$, $N = \tilde{N} = 1$, $l_R = l_L = 0$, $S_R = 1, S_L = -1$ (which corresponds to a ‘massless’ scalar with a Kaluza-Klein charge) is (cf. (5.2)) $M^2 = p_y(p_y - 2\alpha - 2\alpha'\alpha^2 p_y)$. For large enough R , M^2 becomes negative when $\alpha > \alpha_{cr} \sim \frac{1}{2} p_y$. For these states $\gamma = 2\alpha'\alpha p_y$ and thus $\gamma < 2$ if $\alpha > \alpha_{cr}$ and $\alpha' p_y^2 < 2$. The critical value of the magnetic field goes to zero as $R \rightarrow \infty$.^k The $a = \sqrt{3}$ Melvin model is stable in the non-winding ($w = 0$, i.e. $\gamma = 0$) sector, in the sense that it has no new instabilities in addition to the usual flat space tachyon. For $w \neq 0$ (and $0 < \gamma < 2$) there exists a range of parameters q , R for which there is again the same linear instability as in the $a = 1$ Melvin model¹⁶.

5.2. String partition function on the torus

Given the explicit expressions for the Virasoro operators in (4.14), (4.15) it is straightforward to compute the partition function of this conformal model,

$$Z = \int \frac{d^2 \tau}{\tau_2} \int dE \prod_{a=3}^{24} dp_a \sum_{m, w=-\infty}^{\infty} \text{Tr} \exp [2\pi i (\tau \hat{L}_0 - \bar{\tau} \hat{\tilde{L}}_0)]. \quad (5.6)$$

After the integration over the energy, momenta, Poisson resummation and introduction of two auxiliary variables $\lambda, \bar{\lambda}$ (in order to ‘split’ the $O(J^2)$ terms in $\hat{L}_0, \hat{\tilde{L}}_0$ to be able to compute the trace over the oscillators) one finds¹⁶

$$Z(r, \alpha, \beta, q) = \int [d^2 \tau]_1 W(r, \alpha, \beta, q | \tau, \bar{\tau}), \quad (5.7)$$

^k In the noncompact case p_y becomes a continuous parameter representing the momentum of the ‘massless’ state in the y -direction. Thus the ‘massless’ state with an infinitesimal momentum p_y becomes tachyonic for an infinitesimal value of α .

$$[d^2\tau]_1 \equiv d^2\tau \tau_2^{-14} e^{4\pi\tau_2} |f_0(e^{2\pi i\tau})|^{-48} ,$$

where W is given by the sum over windings and the two auxiliary ordinary integrals

$$W = \frac{r}{\alpha'\alpha\beta\tau_2} \sum_{w,w'=-\infty}^{\infty} \int d\lambda d\bar{\lambda} e^{-\mathcal{I}(\chi, \tilde{\chi}, w, w', \tau, \bar{\tau})} \frac{\chi \tilde{\chi} |\theta'_1(0|\tau)|^2}{\theta_1(\chi|\tau) \theta_1(\tilde{\chi}|\bar{\tau})} , \quad (5.8)$$

$$\begin{aligned} \mathcal{I} = & \frac{\pi}{\alpha'\alpha\beta\tau_2} [\chi \tilde{\chi} + \sqrt{\alpha'} r (q + \beta) (w' - \tau w) \tilde{\chi} + \sqrt{\alpha'} r (q - \alpha) (w' - \bar{\tau} w) \chi \\ & + \alpha' r^2 q (q + \beta - \alpha) (w' - \tau w) (w' - \bar{\tau} w) + \frac{1}{2} \alpha' \alpha \beta (\chi - \tilde{\chi})^2] , \\ \chi \equiv & -\sqrt{\alpha'} [2\beta\lambda + qr(w' - \tau w)] , \quad \tilde{\chi} \equiv -\sqrt{\alpha'} [2\alpha\bar{\lambda} + qr(w' - \bar{\tau} w)] . \end{aligned}$$

Z in (5.7),(5.8) can be also obtained by directly computing the string path integral with the action (3.5). The special ‘null’ structure of (3.5) makes possible to compute this non-gaussian path integral exactly (up to the two remaining ordinary integrals over $\lambda, \bar{\lambda}$).

Like the measure in (5.7) W is $SL(2, Z)$ modular invariant (to show this one needs to shift w, w' and redefine $\chi, \tilde{\chi}$). $Z(r, \alpha, \beta, q)$ has several symmetry properties: $Z(r, \alpha, \beta, q) = Z(r, -\beta, -\alpha, q) = Z(r, -\alpha, -\beta, -q) = Z(r, \beta, \alpha, -q)$. It is also invariant under the duality in y direction which transforms the theory with y -period $2\pi R$ and parameters (α, β, q) into the theory with y -period $2\pi\alpha'/R$ and parameters $(q, \beta - \alpha + q, \alpha)$ or parameters $(\alpha - \beta - q, -q, -\beta)$, i.e.

$$Z(r, \alpha, \beta, q) = Z(r^{-1}, q, \beta - \alpha + q, \alpha) = Z(r^{-1}, \alpha - \beta - q, -q, -\beta).$$

For $\alpha = q$ or $\beta = -q$ these relations take their standard ‘circle’ form: $Z(r, \alpha, \beta, \alpha) = Z(r^{-1}, \alpha, \beta, \alpha)$, $Z(r, \alpha, \beta, -\beta) = Z(r^{-1}, \alpha, \beta, -\beta)$. When $\alpha = \beta = q = 0$ the partition function (5.7) is that of the free string compactified on a circle of radius $R = \sqrt{\alpha'} r$. Taking the limit of the non-compact y -dimension ($R \rightarrow \infty$) for generic (α, β, q) one finds that Z (5.7),(5.8) reduces to the partition function of the free bosonic closed string theory.^{*l*}

The expression for Z simplifies when at least one of the parameters α, β, q or $q + \beta - \alpha$ vanishes so that the integrals over $\lambda, \bar{\lambda}$ can be computed explicitly. For example, in the case when either α or β is equal to zero (which includes the constant magnetic field model and $a = \sqrt{3}$ Melvin model) one finds

$$W = r \sum_{w,w'=-\infty}^{\infty} \exp \left(-\frac{\pi}{\tau_2} [r^2 (w' - \tau w) (w' - \bar{\tau} w) + \frac{1}{2} (\chi_0 - \tilde{\chi}_0)^2] \right) \frac{\chi_0 \tilde{\chi}_0 |\theta'_1(0|\tau)|^2}{\theta_1(\chi_0|\tau) \theta_1(\tilde{\chi}_0|\bar{\tau})} ,$$

^{*l*} This generalizes a similar observation for the $\alpha = q = 0$ model¹⁴. In the limit $R = \infty$ the $\alpha = q = 0$ model (1.1) is equivalent to the model of ref.27 which has trivial (free) partition function³¹.

$$\chi_0 = \sqrt{\alpha'}(q + \beta)r(w' - \tau w) , \quad \tilde{\chi}_0 = \sqrt{\alpha'}(q - \alpha)r(w' - \bar{\tau}w) , \quad \alpha\beta = 0 .$$

The magnetic instability of these models (the presence of tachyons in the spectrum) is reflected in singularities (or imaginary parts) of Z . The partition function has new divergences at critical values of the magnetic field parameters when the energy develops an imaginary part.

6. Concluding remarks

The actions (1.1)–(1.3),(3.5) admit straightforward $(1, 1)$ and $(0, 1)$ (or $(1, 0)$) supersymmetric generalizations describing closed superstring and heterotic string models where the two abelian magnetic fields appear in the Kaluza-Klein sector. In addition, it is possible to construct the heterotic string versions of (1.1) and (1.2) which correspond to the same background fields (2.8) and (2.9) but now the magnetic field^m is embedded in the gauge sector of the heterotic string.^{14,15} The idea is to ‘fermionize’ the internal bosonic coordinate y . The non-trivial part of the action of the resulting heterotic string analog of (1.1),(2.8) is¹⁴

$$\begin{aligned} I_{(0,1)} = & \frac{1}{\pi\alpha'} \int d^2\sigma \left[-\partial t \bar{\partial} t - 2A_i \partial t \bar{\partial} x^i + (\delta_{ij} - A_i A_j) \partial x^i \bar{\partial} x^j \right. \\ & - \lambda_L^{\hat{i}} \partial \lambda_L^{\hat{i}} + \lambda_{L i} \partial \lambda_L^i + F_{ij} \partial t \lambda_L^i \lambda_L^j + \frac{1}{2} F_{ij} A_k \partial x^k \lambda_L^i \lambda_L^j \\ & \left. + \bar{\psi}_R (\bar{\partial} - ie_0 A_i \bar{\partial} x^i) \psi_R + \frac{1}{2} ie_0 F_{ij} \bar{\psi}_R \psi_R \lambda_L^i \lambda_L^j \right] , \quad e_0 \equiv R^{-1} = \sqrt{2/\alpha'} , \end{aligned}$$

where $A_i = -\frac{1}{2}\beta\epsilon_{ij}x^j$. Like the bosonic model (1.1) and its direct supersymmetrisations this model can be solved explicitly⁴¹. Detailed study of the pattern of the spectrum in this and similar models (in particular, the special cases when certain higher spin states become massless before becoming tachyonic) may teach us about possible hidden string symmetries. Another interesting direction seems to consider these models in the context of electro-magnetic duality.

Let us mention also that there was a suggestion^{19,21,42} to interpret the Melvin-type solutions of the higher dimensional (dilaton) Einstein-Maxwell theory as alternatives to the standard Kaluza-Klein compactification on compact spaces. The idea was to consider the (ρ, φ) part of the Melvin space (cf.(2.9),(2.10)) as an internal one. Though this 2-space is non-compact, it is ‘nearly closed’ and the corresponding scalar Laplacian has discrete branch in the spectrum (cf.(5.3),(5.2)). The conformal Melvin models (1.2),(1.3) may be used in an attempt of string-theoretic implementation of this idea of having a non-compact space as an internal one (in string models (1.2),(1.3) the internal space is 3-dimensional

^m Note that the two vector fields \mathcal{A} and \mathcal{B} in (2.8),(2.9) are the same up to sign.

(ρ, φ, y)). Since the spectrum of the string mass operator for the Melvin model is explicitly computable, this makes possible to determine the corresponding masses of particles moving in extra flat spatial dimensions. Unfortunately, as in the case of the particle theory limit⁴², this idea does not actually work in the Melvin model: though most of the states in the spectrum belong to its discrete branch, there are also special “zero mode” states (e.g., scalar state with zero charge and orbital momentum in (5.2)) which have continuous mass parameter. It may happen, however, that there are related string models which (like modifications of the Melvin solution discussed in ref.42) may not have this deficiency and yet be explicitly *solvable*. Such models could be also of interest in the context of possible supersymmetry breaking by magnetic backgrounds in internal dimensions (see ref.43 and refs. there).

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